Math 4200
Wednesday September 16

1.6 differentiation and mapping of elementary functions and branches of their inverses, and compositions of all of these.

Announcements: We'll begin by covering the part of Monday's notes which introduces section 1.6, before proceding into today's notes which discuss how to find *branched domains* (aka *fundamental domains*) on which multi-valued functions can be defined as single-valued analytic functions.

Branches of analytic functions overview: If f is *entire,* i.e. analytic on all of $\mathbb{C}$, then it turns out (Picard's Theorem) that if f is *not* a constant function, then the range of f omits no more than two points in $\mathbb{C}$! Furthermore, it turns out that the zeroes of $f'(z)$ are isolated (i.e. if $f'(z_0) = 0$ then there exists $r > 0$ such that $f'(z) \neq 0 \ \forall\ w \in \mathrm{D}(z_0, r) \setminus \{z_0\}$.) So f has a local inverse function except at possibly a countable set of $z \in \mathbb{C}$.

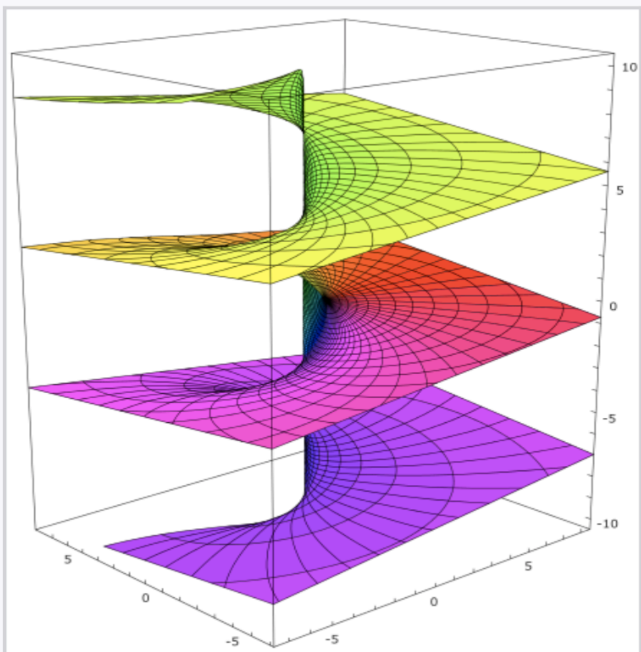
In most cases this means one can construct a differentiable partial "inverse" function g on a *very large* subdomain of $\mathbb{C}$. It will satisfy half of the inverse function condition, namely

$$f(g(z)) = z.$$

And the domain of g can usually be chosen to be a connected open domain $A \subseteq \mathbb{C}$ with a just finite number of curves removed from $\mathbb{C}$ to get A. These omitted curves are called *branch cuts*, and the choice of (partial) inverse function is called a *branch* of the inverse function. Branch cuts always terminate either at ∞ (which means $|z| \to \infty$), or at finite points, and these are called *branch points.*

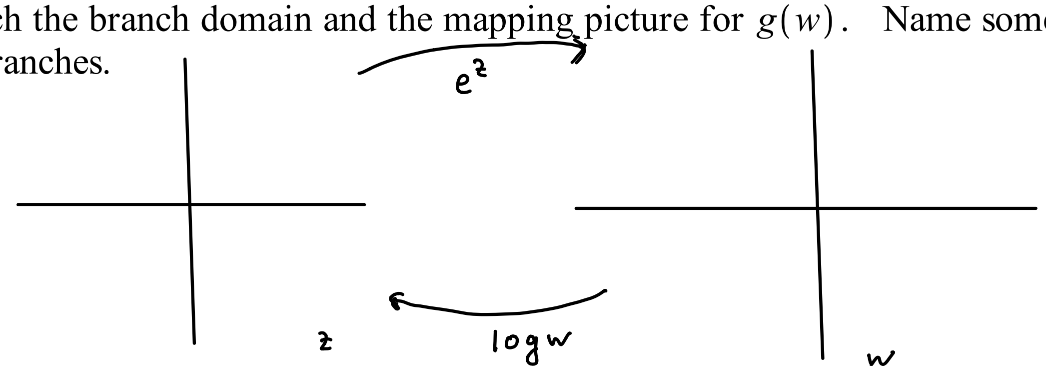
In our text section 1.6 these branch domains are called *fundamental domains*. There is usually some freedom in how they are chosen.

The most central example of this discussion is $f(z) = e^z$ which omits only the point 0 in its range, and branch choices for the multivalued inverse $\log(z)$. A nice graphic picture from the wikipedia page on the complex logarithm which visualizes the possible branch choices for $\log(z)$, is obtained by plotting the parametric surface $(r\cos(\theta), r\sin(\theta), \theta)$ in $\mathbb{R}^3$. I haven't figured out precisely what the curves on the helicoid are, although they seem to be related to some conformal parameterization of the helicoid, not the $r-\theta$ one. Since the helicoid is a *minimal surface*, i.e. locally area minimizing and a possible shape for soap films, it turns out that it can be parameterized in a conformal way using harmonic functions. (!)



A plot of the multi-valued imaginary part of the complex logarithm function, which shows the branches. As a complex number *z* goes around the origin, the imaginary part of the logarithm goes up or down. This makes the origin a branch point of the function.

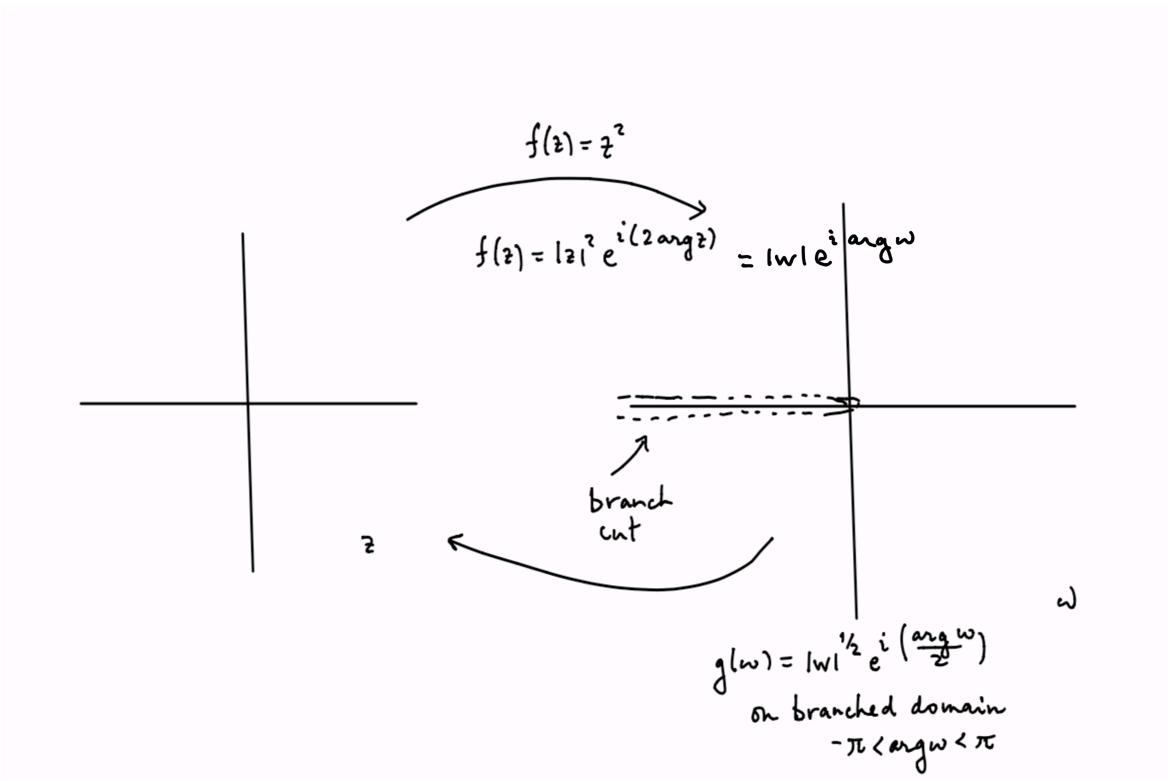
Example 1) $f(z) = e^z$, $g(w) = \log w = \ln|w| + i\, arg\, w$ where we choose $-\pi < arg(w) < \pi$. This is called the standard branch of $\log w$. We've seen this before, but sketch the branch domain and the mapping picture for $g(w)$. Name some other possible branches.



Example 2) $f(z)=z^2$, $g(w)=\sqrt{w}$ (for some branch choice). Note for any branch choice of g,

$$f(g(w))=w$$
$$f'(g(w))g'(w)=1$$
$$g'(w)=\frac{1}{f'(g(w))}=\frac{1}{2\,g(w)}=\frac{1}{2}w^{-\frac{1}{2}}.$$

Describe the range of the branch of the square root function defined below. Write down two other branch choices - one using the same branch cut, and another one using a different cut.



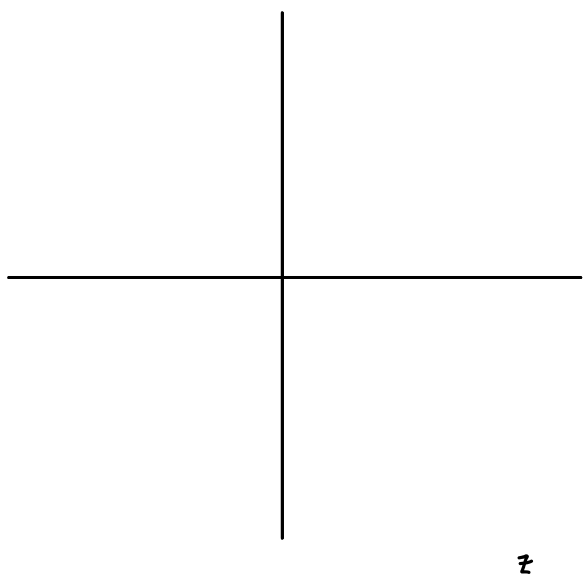
Example 3) Find a definition and branched domain for

$$f(z) = \sqrt{z^2 - 1}\,.$$

(In your homework for next week you will do an analogous procedure for $g(z) = \sqrt{z^3 - 1}$.) Begin by identifying branch points based on where f or f' cannot be not defined as an analytic function.

Then

a) Writing $f(z) = \sqrt{z^2 - 1} = \sqrt{z - 1}\,\sqrt{z + 1}$ leads to one possible way of proceding.



b) Considering f as a composition, $f(z) = g \circ h(z)$ with $h(z) = z^2 - 1$ and $g(w) = \sqrt{w}$ recovers the first branched domain, but also leads to a choice with only a finite branch cut, as well as the original one.